Category Theory
Introductory Notes

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Contents

1 Categories and diagrams

Definition 1.1 (Quiver)**.** A *quiver*[1] is a directed graph that may have multiple arrows with the same source and target vertices. A quiver $\mathcal{J}$ consists of:

1. A collection of vertices $|\mathcal{J}|$.

   Individual vertices are denoted by A, B, X, Y, Z, where $A\colon\mathcal{J}$ stresses that A is in $|\mathcal{J}|$.
2. For every pair of vertices X, Y a family of arrows $\mathcal{J}[X, Y]$.

   Individual arrows f, g, h in $\mathcal{J}[X, Y]$ are denoted by $f\colon X \to Y$. $\Diamond$

Quivers are depicted graphically as dots connected by arrows where a dot represents exactly one vertex and a depicted arrow represents exactly one arrow of the quiver.

$$\begin{tikzcd} \bullet & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

Definition 1.2 (Category)**.** A *category* is a quiver with additional structure. In a category, vertices X, Y are called objects and arrows $f\colon X \to Y$ are called morphisms. A category $\mathcal{C}$ is a quiver along with:

3. For every object $A\colon\mathcal{C}$ a designated identity morphism $\mathrm{id}_A\colon A \to A$.
4. A composition operator $\_ \circ \_$ that maps a pair of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ to a morphism $(g \circ f)\colon X \to Z$ such that the following conditions are satisfied.

$$\mathrm{id}_Z \circ g \;=\; g \qquad (f \circ g) \circ h \;=\; f \circ (g \circ h) \qquad f \circ \mathrm{id}_X \;=\; f \qquad \Diamond$$

For a morphism $f\colon A \to B$ we call A the domain of f and B the codomain of f. The collection of morphisms of $\mathcal{C}$ with domain A is denoted by $\mathcal{C}[A, \_]$ and the collection of morphisms with codomain B is denoted by $\mathcal{C}[\_, B]$. The collection of all morphisms of $\mathcal{C}$ is denoted by $\mathcal{C}[\_, \_]$.

Quivers that do not contain any arrows are said to be *discrete*. Likewise, a discrete category is a category that has no morphisms other than the identity morphisms.

Definition 1.3 (Diagram)**.** A *diagram* is a labelled quiver. Specifically, if $\mathcal{J}$ is a quiver then a diagram *of shape* $\mathcal{J}$ *in a category* $\mathcal{C}$ is a labelling $L\colon\mathcal{J} \to \mathcal{C}$ which assigns to every vertex $A\colon\mathcal{J}$ an object $(LA)\colon\mathcal{C}$ and to every arrow $f\colon A \to B$ of $\mathcal{J}$ a morphism $Lf\colon LA \to LB$ of $\mathcal{C}$. $\Diamond$

Intuitively it is clear that L is a labelled quiver if one thinks of L as a set of ordered pairs of type (A, LA) and (f, Lf). As one would expect, diagrams in a category $\mathcal{C}$ are depicted as quivers with labels added to the dots and arrows. As an example let X, Y, Z be objects of $\mathcal{C}$ and let $f\colon X \to Y$, $h\colon Z \to Y$ and $g\colon X \to Y$ be morphisms of $\mathcal{C}$. Below are examples of diagrams in $\mathcal{C}$ based on the quivers depicted previously.

$$\begin{tikzcd} X & Y \end{tikzcd} \qquad \begin{tikzcd} & Z \arrow[d, "h"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Y \\ Y & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

[1] In real life, A quiver is a container of arrows, typically carried by archers.

Definition 1.4 (Commutative diagram)**.** A diagram $L\colon \mathcal{J} \to \mathcal{C}$ is *commutative* if for any two paths with the same start and end object of $\mathcal{J}$, the labels on arrows compose to the same morphism of $\mathcal{C}$.

$$\text{For all} \quad LA \xrightarrow{Lf_1} LX \xrightarrow{Lf_2} \cdots \xrightarrow{Lf_n} LB \quad \text{and} \quad LA \xrightarrow{Lg_1} LY \xrightarrow{Lg_2} \cdots \xrightarrow{Lg_m} LB,$$
$$LA \xrightarrow{Lf_n \circ \ldots \circ Lf_2 \circ Lf_1} LB \quad = \quad LA \xrightarrow{Lg_m \circ \ldots \circ Lg_2 \circ Lg_1} LB. \qquad \Diamond$$

A few words on the n and m–ary compositions above are required. For any composition $f_1 \circ \ldots \circ f_n$ we have that $f_1 \circ \ldots \circ f_n = \mathrm{id}_X \circ f_1 \circ \ldots \circ f_n \circ \mathrm{id}_Y$ for suitable X and Y. Whenever we specify a morphism as a composition $(f_1 \circ \ldots \circ f_n)\colon X \to Y$ we define the case $n = 0$ to be id_X in which case, of course, $X = Y$.

As an example of commutative diagrams, note that in the rightmost diagram depicted previously, to state that it commutes implies that $f = g$ whereas this is not the case in the second rightmost diagram.

Given a quiver $\mathcal{J}$ and a category $\mathcal{C}$, one particular family of commutative diagrams is the family of diagonal diagrams.

Definition 1.5 (Diagonal diagram)**.** For an object $A\colon\mathcal{C}$ the *diagonal diagram* $\Delta_A\colon \mathcal{J} \to \mathcal{C}$ of shape $\mathcal{J}$ assigns to any vertex of $\mathcal{J}$ the object $A\colon\mathcal{C}$ and to any arrow of $\mathcal{J}$ the morphism id_A of $\mathcal{C}$. $\Diamond$

In some sense, a diagonal diagram represents just an object. The fact that it can be of any shape however, makes them a useful tool and we will soon see an example of this.

2 Morphisms

Definition 2.1 (Monomorphism and epimorphism)**.**

1. A *monomorphism* $m\colon Y \rightarrowtail Z$ is a morphism $m\colon Y \to Z$ such that for any pair of morphisms $g\colon X \to Y$ and $h\colon X \to Y$, if $m \circ g = m \circ h$ then $g = h$. In diagrams, if the diagram on the left commutes, then so does the diagram on the right.

$$\begin{tikzcd} & Y \arrow[dr, "m"] & \\ X \arrow[ur, "g"] \arrow[dr, "h"'] & & Z \\ & Y \arrow[ur, "m"'] & \end{tikzcd} \qquad \Longrightarrow \qquad \begin{tikzcd} X \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Y \end{tikzcd}$$

2. An *epimorphism* $e\colon X \twoheadrightarrow Y$ is a morphism $e\colon X \to Y$ such that for any pair $g\colon Y \to Z$ and $h\colon Y \to Z$, if $g \circ e = h \circ e$ then $g = h$.

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "e"] \arrow[dr, "e"'] & & Z \\ & Y \arrow[ur, "h"'] & \end{tikzcd} \qquad \Longrightarrow \qquad \begin{tikzcd} Y \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Z \end{tikzcd}$$

$\Diamond$

Identity morphisms are monomorphisms. If $\mathrm{id} \circ g = \mathrm{id} \circ h$ then obviously $g = h$. Likewise, if $g \circ \mathrm{id} = h \circ \mathrm{id}$ then $g = h$ thus identity morphisms are epimorphisms, too Moreover, monomorphisms of any category $\mathcal{C}$ are closed under composition, as are epimorphisms. For let $e_1 : X \twoheadrightarrow Y$ and $e_2 \colon Y \twoheadrightarrow Z$ be epimorphisms. Consider two arbitrary morphisms $f : Z \to A$ and $g : Z \to A$ and assume that $f \circ e_2 \circ e_1 = g \circ e_2 \circ e_1$. Then $f \circ e_2 = g \circ e_2$ because e_1 is an epimorphism and subsequently $f = g$ because e_2 is an epimorphism. Therefore $e_2 \circ e_1$ is an epimorphism. With this in mind it makes sense to define the categories $\mathrm{Mono}\mathcal{C}$ and $\mathrm{Epi}\mathcal{C}$ as follows.

Definition 2.2 (Category of monos, epis)**.** Let $\mathcal{C}$ be a category.

1. The category $\mathrm{Mono}\mathcal{C}$ consists of all objects of $\mathcal{C}$ and all monomorphisms between them.

   Identities and compositions are defined as in $\mathcal{C}$.
2. The category $\mathrm{Epi}\mathcal{C}$ consists of all objects of $\mathcal{C}$ and all epimorphisms between them.

   Identities and compositions are defined as in $\mathcal{C}$. $\Diamond$

Recall that identity morphisms are monomorphisms. Another instance of a monomorphism is a morphism $m\colon X \to Y$ that can be 'undone' by a morphism $e\colon Y \to X$. If this is the case then we call e a retraction of m and we call m a section of e.

Definition 2.3 (Section and retraction)**.** Let $m\colon X \to Y$ and $e\colon Y \to X$. If $e \circ m = \mathrm{id}_X$ then e is a *retraction* of m and m is a *section* of e.

$$\begin{tikzcd} X \arrow[rr, "\mathrm{id}_X"] \arrow[dr, "m"'] & & X \\ & Y \arrow[ur, "e"'] & \end{tikzcd}$$

$\Diamond$

You may be inclined to think that the direction of the identity arrow in the diagram above is irrelevant. This is certainly not the case. Asserting that the diagram commutes with the id_X

arrow in the converse direction does indeed assert that $e \circ m = \mathrm{id}_X$ but it also asserts that $m \circ e = \mathrm{id}_Y$ which is *not* implied by commutativity of the original diagram.

A morphism $m : X \to Y$ that has a retraction $e : Y \to X$ is certainly a monomorphism. Consider two morphisms $g : A \to X$ and $h : A \to X$ and assume that $m \circ g = m \circ h$. Then $e \circ m \circ g = e \circ m \circ h$ and because e is a retraction of m we can conclude that $g = h$ which confirms that m is a monomorphism. The dual of this argument shows that any morphism that has a section is an epimorphism.

A more specific case occurs if a morphism $f : X \to Y$ has a retraction $g : Y \to X$ but in addition $f : X \to Y$ is a retraction of $g : Y \to X$. If this is the case then f and g are isomorphisms.

Definition 2.4 (Isomorphism)**.** Let $f : X \to Y$ and $g : Y \to X$. Then g is the *inverse* of f and f is the inverse of g if $g \circ f = \mathrm{id}_X$ *and* $f \circ g = \mathrm{id}_Y$. An *isomorphism* $f : X \xrightarrow{\sim} Y$ is a morphism that has an inverse.

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

$\Diamond$

The inverse of an isomorphism $f : X \xrightarrow{\sim} Y$ is easily seen to be unique and is denoted by $f^{-1} : Y \xrightarrow{\sim} X$. An isomorphism is both a section and a retraction of its inverse therefore it is both a monomorphism and an epimorphism.

So far all properties of morphisms that we have discussed have been defined in terms of commutativity. There has been no mention of an internal structure of objects or morphisms. In category theory properties are commonly defined in this way. The most common category in which objects and morphisms do have an 'inner structure' is the category Set of sets and total functions. Properties of sets and functions often are expressed by referring to their internal structure. Typical examples are injectivity and surjectivity, which are defined by referring to the elements of the domain and codomain of a function. In Set the definitions of monomorphism and epimorphism provide an intentional alternative. A morphism $f : X \to Y$ of Set is injective only if it is a monomorphism, and it is surjective only if it is an epimorphism. The inner structure on sets suggests a particular kind of monomorphism, one that cannot be defined intensionally.

Definition 2.5 (Inclusion)**.** In the category Set an *inclusion* $f : X \hookrightarrow Y$ is a morphism $f : X \rightarrowtail Y$ such that $f(x) = x$ for all $x \in X$. $\Diamond$

3 Functors

Definition 3.1 (Functor)**.** A *functor* $F\colon\mathcal{C} \to \mathcal{D}$ is a map that takes objects $A\colon\mathcal{C}$ to objects $(FA)\colon\mathcal{D}$ and morphisms $f\colon A \to B$ to morphisms $Ff\colon FA \to FB$ whilst preserving identities and compositions.

$$F\mathrm{id}_X = \mathrm{id}_{FX} \qquad\qquad F(f \circ g) = Ff \circ Fg \qquad\qquad \Diamond$$

A functor $F\colon\mathcal{C} \to \mathcal{D}$ is *faithful* if for all X, Y of $\mathcal{C}$ the map $F\colon\mathcal{C}[X, Y] \to \mathcal{D}[FX, FY]$ is injective and F is *full* if for all X, Y of $\mathcal{C}$ the map $F\colon\mathcal{C}[X, Y] \to \mathcal{D}[FX, FY]$ is surjective. A category $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$ if there is a faithful inclusion functor $I\colon\mathcal{D} \hookrightarrow \mathcal{C}$. If I is both full and faithful then $\mathcal{D}$ is a *full subcategory* of $\mathcal{C}$.
As the reader will have noticed, for an object A we write functor application as FA rather than the classical $F(A)$. Given a suitable functor G, functor composition will be written as GF. This does not cause any ambiguity: the notation GFA might be read as $G(FA)$ and $(GF)A$ which is $G(F(A))$ and $(G \circ F)(A)$ in classical notation. Finally, we shall use the notation $F^n A$ for iterated applications of F, thus $F^0 A \coloneqq A$ and $F^{m+1} A \coloneqq FF^m A$.

3.1 Containers

Intuitively one may think of a functor as a container type. If X and Y are collections of elements then FX and FY are collections of containers, where a container in FX contains elements of X and a container in FY contains elements of Y. Given a morphism $f\colon X \to Y$ the morphism $Ff\colon FX \to FY$ takes a container in FX to a container in FY by applying f to every contained element.

3.2 Preservation of commutative diagrams

A functor is a map between categories that preserves commutative diagrams. Let $F\colon\mathcal{C} \to \mathcal{D}$ be a functor and let $L\colon\mathcal{J} \to \mathcal{C}$ be a diagram of $\mathcal{C}$. Since F takes objects to objects and morphisms to morphisms we can consider what one might call the image of L under F which contains a labelled vertex (A, FLA) for ever labelled vertex (A, LA) of L and a labelled arrow (f, FLf) for every (f, Lf) of L. Hence, we define the image of a diagram $L\colon\mathcal{J} \to \mathcal{C}$ under a functor $F\colon\mathcal{C} \to \mathcal{D}$ to be the diagram $FL\colon\mathcal{J} \to \mathcal{D}$.

Now suppose that the diagram $L\colon\mathcal{J} \to \mathcal{C}$ commutes. Then its image $FL\colon\mathcal{J} \to \mathcal{D}$ commutes as well. For consider any two paths between LA and LB of L consisting of labelled arrows $Lf_1, \ldots, Lf_n$ and $Lg_1, \ldots, Lg_m$. Their labels compose to the same morphism $Lf_n \circ \ldots \circ Lf_1 = Lg_m \circ \ldots \circ Lg_1$ of $\mathcal{C}$ by definition of commutativity.

$$\begin{tikzcd}
\mathcal{C} & \cdots \arrow[dr, "Lf_n"] & \\
LA \arrow[ur, "Lf_1"] \arrow[dr, "Lg_1"'] \arrow[rr, bend left=60, "Lf_n \circ \ldots \circ Lf_1"] \arrow[rr, bend right=60, "Lg_m \circ \ldots \circ Lg_1"'] & & LB \\
& \cdots \arrow[ur, "Lg_m"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \cdots \arrow[dr, "FLf_n"] & \mathcal{D} \\
FLA \arrow[ur, "FLf_1"] \arrow[dr, "FLg_1"'] \arrow[rr, bend left=60, "F(Lf_n \circ \ldots \circ Lf_1)"] \arrow[rr, bend right=60, "F(Lg_m \circ \ldots \circ Lg_1)"'] & & FLB \\
& \cdots \arrow[ur, "FLg_m"'] &
\end{tikzcd}$$

Consider the image of these morphisms under F. Since F is a functor it preserves composition.

$$FLf_n \circ \ldots \circ FLf_1 \quad = \quad F(Lf_n \circ \ldots \circ Lf_1) \quad = \quad F(Lg_m \circ \ldots \circ Lg_1) \quad = \quad FLg_m \circ \ldots \circ FLg_1$$

Thus, any two sequences of labelled arrows of FL compose to the same morphism of $\mathcal{D}$ which shows that the diagram $FL\colon \mathcal{J} \to \mathcal{D}$ commutes.

4 Category shaped diagrams

Consider a quiver $\mathcal{J}$ and a diagram $L\colon \mathcal{J} \to \mathcal{C}$. Since categories are quivers with additional structure, $\mathcal{J}$ might be a category. If in addition, L preserves identities and compositions then $L\colon \mathcal{J} \to \mathcal{C}$ is a functor and L is called a category–shaped diagram. Quiver–shaped diagrams are more intuitive than category–shaped diagrams and the graphical diagrams that are common in category theory are depictions of quiver–shaped diagrams. However, category–shaped diagrams are often easier to work with because they get rid of the distinction between diagrams and functors. Fortunately, any quiver–shaped diagram can be presented as a category–shaped diagram in a way that preserves their commutativity, a construction that will be shown in a minute.

Definition 4.1 (Free category)**.** Given a quiver $\mathcal{J}$, the free category $\mathcal{K}$ based on $\mathcal{J}$ is constructed as follows.

- Any vertex $A\colon \mathcal{J}$ is an object $A\colon \mathcal{K}$.
- Any path $A \xrightarrow{f_1} B \xrightarrow{f_2} \ldots \xrightarrow{f_n} X$ in $\mathcal{J}$ of length $n \geq 0$ is a morphism of $\mathcal{K}[A, X]$.
- For any object $A\colon \mathcal{K}$ the morphism id_A is the zero–length path from $A\colon \mathcal{J}$ to $A\colon \mathcal{J}$.
- Composition of morphisms is defined as path concatenation:

$$C \xrightarrow{f_1} \cdots \xrightarrow{f_n} X \quad \circ \quad A \xrightarrow{g_1} \cdots \xrightarrow{g_m} C \quad := \quad A \xrightarrow{g_1} \cdots \xrightarrow{g_m} C \xrightarrow{f_1} \cdots \xrightarrow{f_n} X. \qquad \Diamond$$

The free category construction allows us to present any quiver–shaped diagram as a category–shaped diagram. Let $L\colon \mathcal{J} \to \mathcal{C}$ be a quiver–shaped diagram and let $\mathcal{K}$ be the free category on $\mathcal{J}$. Construct a category–shaped diagram $D\colon \mathcal{K} \to \mathcal{C}$ by putting:

$$\begin{aligned} DX &:= LX \\ D(X \xrightarrow{f_1} \cdots \xrightarrow{f_n} Y) &:= Lf_n \circ \ldots \circ Lf_1 \end{aligned}$$

We would like to use D as a 'drop in' replacement for L. In particular we would like D to commute only if L commutes.

So assume that D does indeed commute. Consider any two paths between some pair of vertices A and B of $\mathcal{J}$. They are morphisms $A \xrightarrow{f_1} \cdots \xrightarrow{f_n} B$ and $A \xrightarrow{g_1} \cdots \xrightarrow{g_m} B$ of $\mathcal{K}$. By commutativity, D maps these morphisms of $\mathcal{K}$ to the same morphism of $\mathcal{C}$ and by definition of D they get mapped to $Lf_n \circ \ldots \circ Lf_1 = Lg_m \circ \ldots \circ Lg_1$ which confirms that L commutes.

Conversely, assume that D does not commute, as witnessed by two paths in $\mathcal{K}$. These paths compose to two parallel morphisms of $\mathcal{K}$. The paths in $\mathcal{K}$ labelled by D compose to distinct morphisms of $\mathcal{C}$. Since D preserves composition it maps the parallel morphisms of $\mathcal{K}$ to the same two distinct morphisms of $\mathcal{C}$. Let the parallel morphisms of $\mathcal{K}$ be $A \xrightarrow{f_1} \cdots \xrightarrow{f_n} B$ and $A \xrightarrow{g_1} \cdots \xrightarrow{g_m} B$. Note that they are paths in $\mathcal{J}$. By definition of D they get mapped to $Lf_n \circ \ldots \circ Lf_1 \neq Lg_m \circ \ldots \circ Lg_1$ which confirms that L does not commute.

5 Functor categories

Definition 5.1 (Natural transformation)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\nu : F \xrightarrow{\cdot} G$ is a family of morphisms $(\nu_X : FX \to GX)_{X : \mathcal{C}}$ such that for all morphisms $f : X \to Y$ of $\mathcal{C}$ the following diagram in $\mathcal{D}$ commutes.

$$\begin{tikzcd} FX \arrow[r, "\nu_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\nu_Y"'] & GY \end{tikzcd}$$

$\lozenge$

The diagram above is called a *naturality square* for ν and a morphism $\nu_X : FX \to GX$ is called the *component* of ν at $X : \mathcal{C}$.

Definition 5.2 (Functor category)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *functor category* $\mathcal{D}^{\mathcal{C}}$ is defined as follows.

- Every functor $F : \mathcal{C} \to \mathcal{D}$ is an object of $\mathcal{D}^{\mathcal{C}}$.
- Every natural transformation $\nu : F \xrightarrow{\cdot} G$ is a morphism from F to G.
- For every object F the identity morphism $\mathrm{id}_F : F \xrightarrow{\cdot} F$ is the natural transformation $(\mathrm{id}_{FX} : FX \to FX)_{X : \mathcal{C}}$, consisting of identity morphisms of $\mathcal{D}$.
- For any two morphisms $\nu : F \xrightarrow{\cdot} H$ and $\mu : H \xrightarrow{\cdot} G$ the composition $(\mu \circ \nu) : F \xrightarrow{\cdot} G$ is the natural transformation $(\mu_X \circ \nu_X : FX \to GX)_{X : \mathcal{C}}$. $\lozenge$

It is not hard to check that compositions and identities are well defined. In particular, for all morphisms $f : X \to Y$ of $\mathcal{C}$ the diagram on the left commutes because $(\mathrm{id}_F)_X :\equiv \mathrm{id}_{FX}$. The diagram on the right commutes because it consists of two naturality squares.

$$\begin{tikzcd} FX \arrow[r, "(\mathrm{id}_F)_X"] \arrow[d, "Ff"'] & FX \arrow[d, "Ff"] \\ FY \arrow[r, "(\mathrm{id}_F)_Y"'] & FY \end{tikzcd} \qquad \begin{tikzcd} FX \arrow[r, "\nu_X"] \arrow[d, "Ff"'] & HX \arrow[r, "\mu_X"] \arrow[d, "Hf"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\nu_Y"'] & HY \arrow[r, "\mu_Y"'] & GY \end{tikzcd}$$

Since category–shaped diagrams are just functors, we already have an example of a functor category. Let $\mathcal{J}$ and $\mathcal{C}$ be categories. Objects of the functor category $\mathcal{C}^{\mathcal{J}}$ are category–shaped diagrams $L : \mathcal{J} \to \mathcal{C}$. In the following section we shall see an example of natural transformations between diagrams in a functor category.

6 Cones and Limits

An appealing feature of category theory is that it offers a way to formalise an intuitive notion of canonicity. Sometimes mathematical structures are said to arise naturally, in that there appears to be only one natural choice among a class of structures that have the required properties. A way to study this intuitive notion of a canonical structure is to study the mathematical structure of the solution space itself. This is a powerful concept. Design in the general sense, might appear to involve lots of arbitrary design decisions (at least, to the unpracticed eye). In liberal arts, designers are typically guided by intuition. A designer might use some formal guidelines but many decisions are made based on intuition and experience of how to achieve the desired effect. If we are able to uncover the mathematical structure of the solution space then design decisions can be formally described and studied. Often, depending on the structure of the solution space may be able to uncover a best, or canonical solution. The key concept in category theory that formalises this intuitive notion of canonicity are initial and final objects.

Definition 6.1 (Initial object and final object)**.**

- An *initial object* of a category $\mathcal{C}$ is an object A such that for every object $X : \mathcal{C}$ there is a *unique* morphism $\mathsf{i}_X : A \to X$.
- Dually, a *final object* of $\mathcal{C}$ is an object Z such that for every object $X : \mathcal{C}$ there is a *unique* morphism $!_X : X \to Z$. $\Diamond$

Often we will speak of *the* initial– and *the* final object of a category. This is justified by the fact that initial– and final objects are unique up to unique isomorphism. Consider a category $\mathcal{C}$ and two initial objects A and B. By definition of initiality of A and B respectively there are morphisms $\mathsf{i}_B : A \to B$ and $\mathsf{i}_A : B \to A$ and thus a morphism $(\mathsf{i}_A \circ \mathsf{i}_B) : A \to A$. However by initiality the identity morphism $\mathrm{id}_A : A \to A$ is the *unique* morphism from A to A thus $\mathsf{i}_A \circ \mathsf{i}_B = \mathrm{id}_A$. The converse holds for B and as such $\mathsf{i}_B : A \to B$ and $\mathsf{i}_A : B \to A$ are isomorphisms as claimed. For clarity, if we speak of *the* initial object of a category $\mathcal{C}$, we refer to a single representative object of the class of initial objects of $\mathcal{C}$.

Often a category can be constructed in such a way that the 'canonical solution' we wish to describe is exactly the initial– or final object of this category. One particular example is the category of cones over a diagram.

Definition 6.2 (Cones and cocones)**.** Let $L : \mathcal{C}^{\mathcal{J}}$ be a (category–shaped) diagram.

- A *cone* from $A : \mathcal{C}$ to L is a pair (A, φ) where φ is a natural transformation $\varphi : \Delta_A \stackrel{.}{\to} L$.
- A *cocone* from L to $Z : \mathcal{C}$ is a pair (κ, Z) where κ is a natural transformation $\kappa : L \stackrel{.}{\to} \Delta_Z$. $\Diamond$

It is easily checked that the definition above amounts to the following:

- A cone from $A : \mathcal{C}$ to L is a pair (A, φ) where φ is a family of morphisms $(\varphi_X : A \to LX)_{X : \mathcal{J}}$ such that for every $f : X \to Y$ of $\mathcal{J}$ the two leftmost diagrams below, commute.
- A cocone from L to an object $Z : \mathcal{C}$ is a pair (κ, Z) where κ is a family of morphisms $(\kappa_X : LX \to Z)_{X : \mathcal{J}}$ such that for every $f : X \to Y$ of $\mathcal{J}$ the two rightmost diagrams commute.

$$\begin{tikzcd} & A \arrow[dl, "\varphi_X"'] \arrow[dr, "\varphi_Y"] & \\ LX \arrow[rr, "Lf"'] & & LY \end{tikzcd} \qquad \begin{tikzcd} \Delta_A X \arrow[r, "\Delta_A f"] \arrow[d, "\varphi_X"'] & \Delta_A Y \arrow[d, "\varphi_Y"] \\ LX \arrow[r, "Lf"'] & LY \end{tikzcd} \qquad \begin{tikzcd} LX \arrow[r, "Lf"] \arrow[d, "\kappa_X"'] & LY \arrow[d, "\kappa_Y"] \\ \Delta_Z X \arrow[r, "\Delta_Z f"'] & \Delta_Z Y \end{tikzcd} \qquad \begin{tikzcd} LX \arrow[rr, "Lf"] \arrow[dr, "\kappa_X"'] & & LY \arrow[dl, "\kappa_Y"] \\ & Z & \end{tikzcd}$$

Cones and cocones form a category. We define the category of cones, the category of cocones is defined likewise.

Definition 6.3 (Category of cones)**.** Let $L\colon \mathcal{J} \to \mathcal{C}$ be a category–shaped diagram. The *category of cones over* L is defined as follows.

1. Objects are cones $\varphi\colon \Delta_A \dot{\to} L$.
2. For a pair of cones $\varphi\colon \Delta_A \dot{\to} L$ and $\psi\colon \Delta_B \dot{\to} L$ a morphisms from φ to ψ is a morphism $g\colon A \to B$ of $\mathcal{C}$ such that for all X and Y of $\mathcal{J}$ the following commutes.

$$\begin{tikzcd} & A \arrow[d, "g"] \arrow[ddl, bend right, "\varphi_X"'] \arrow[ddr, bend left, "\varphi_Y"] & \\ & B \arrow[dl, "\psi_X"'] \arrow[dr, "\psi_Y"] & \\ LX \arrow[rr, "Lf"'] & & LY \end{tikzcd}$$

3. The identity morphism of a cone $\varphi\colon \Delta_A \dot{\to} L$ is the morphism $\mathrm{id}_A\colon A \to A$ of $\mathcal{C}$.
4. Composition of morphisms is defined as composition of the underlying morphisms of $\mathcal{C}$. ◊

Definition 6.4 (Limits and colimits)**.** Let $L\colon \mathcal{J} \to \mathcal{C}$ be a category–shaped diagram.

- A *limit* of L is a final object $\pi\colon \Delta_A \dot{\to} L$ in the category of cones over L.
- A *colimit* of L is an initial object $\iota\colon L \dot{\to} \Delta_Z$ in the category of cocones over L. ◊

A limit of a diagram $L\colon \mathcal{J} \to \mathcal{C}$ is a pair (A, π) of an object $A\colon \mathcal{C}$ and a family of morphisms $(\pi_X : A \to LX)_{X\,:\,\mathcal{J}}$. The morphisms $\pi_X : A \to LX$ are called *projections*, and the notation π_X is reserved for morphisms of final cones specifically, although we sometimes subscript it with the object LX of $\mathcal{C}$ rather than the vertex X of $\mathcal{J}$. Likewise, a colimit of a a diagram $L\colon \mathcal{J} \to \mathcal{C}$ is a pair (ι, Z) where ι is a family of morphisms $(\iota_X : LX \to Z)_{X\,:\,\mathcal{J}}$ into an object $Z\colon \mathcal{C}$, called *injections*. Again, the notation ι_X, or ι_{LX}, is used specifically for morphisms of initial cocones. Below, the most common limits and colimits are introduced.

Definition 6.5 (Common limits and colimits)**.**

- Let $\mathcal{J}$ be the discrete category with two objects. A diagram $L\colon \mathcal{J} \to \mathcal{C}$ selects two objects X and Y of $\mathcal{C}$. The binary *product* $X \times Y$ is the final cone over L. The unique morphism $h\colon A \to X \times Y$ is denoted by (ψ_X, ψ_Y).

$$\begin{tikzcd} & A \arrow[dl, "\psi_X"'] \arrow[d, dashed, "h"] \arrow[dr, "\psi_Y"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[dr, "\psi_X"'] & X + Y \arrow[d, dashed, "h"] & Y \arrow[l, "\iota_Y"'] \arrow[dl, "\psi_Y"] \\ & Z & \end{tikzcd}$$

- Again, let $\mathcal{J}$ be the discrete category with two objects so that $L\colon \mathcal{J} \to \mathcal{C}$ selects two objects X and Y of $\mathcal{C}$. The binary *coproduct* $X + Y$ is the initial cocone over L. The unique morphism $h\colon X + Y \to Z$ is denoted by $[\psi_X, \psi_Y]$.
- Let $\mathcal{J}$ be the free category over the quiver consisting of two arrows with distinct domains and a common codomain. A diagram $L\colon \mathcal{J} \to \mathcal{C}$ selects three objects and two morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ of $\mathcal{C}$. The *pullback* of f and g is the final cone over L.

$$\begin{tikzcd}
Q \arrow[dr, dashed, "h"] \arrow[drr, bend left, "\psi_Y"] \arrow[ddr, bend right, "\psi_X"'] & & \\
& P \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\iota_Y"] \arrow[ddr, bend left, "\psi_Y"] & \\
X \arrow[r, "\iota_X"'] \arrow[drr, bend right, "\psi_Z"'] & P \arrow[dr, dashed, "h"] & \\
& & Q
\end{tikzcd}$$

- Let $\mathcal{J}$ be the free category over the quiver consisting of two arrows with a common domain but distinct codomains. A diagram $L\colon \mathcal{J} \to \mathcal{C}$ selects three objects and two morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ of $\mathcal{C}$. The *pushout* of f and g is the initial cocone over L. ◊

7 Comma categories

Definition 7.1 (Comma category)**.** Let $\mathcal{D}$, $\mathcal{C}$ and $\mathcal{E}$ be categories and let $F\colon \mathcal{D} \to \mathcal{C}$ and $G\colon \mathcal{E} \to \mathcal{C}$ be functors between them. The comma category $(F{\downarrow}G)$ of F and G is defined as follows.

- Objects of $(F{\downarrow}G)$ are triples (X, f, Y) where X is an object of $\mathcal{D}$, $f\colon FX \to GY$ is a morphism of $\mathcal{C}$ and Y is an object of $\mathcal{E}$.
- A morphism from (X, f, Y) to (X', f', Y') is a pair (g, h) where $g\colon X \to X'$ is a morphism of $\mathcal{D}$ and $h\colon Y \to Y'$ is a morphism of $\mathcal{E}$ such that the following commutes in $\mathcal{C}$.

$$\begin{tikzcd} FX \arrow[r, "Fg"] \arrow[d, "f"'] & FX' \arrow[d, "f'"] \\ GY \arrow[r, "Gh"'] & GY' \end{tikzcd}$$

- The identity morphism $\mathrm{id}_{(X,f,Y)}$ of an object (X, f, Y) is simply pair $(\mathrm{id}_X, \mathrm{id}_Y)$.
- Composition $(g', h') \circ (g, h)$ is defined by $(g' \circ g, h' \circ h)$. $\Diamond$

Definition 7.2 (Slice– and coslice categories)**.** Let $\mathbf{1}$ be the category consisting of a single object and its identity morphism. The diagonal functor $\Delta_X\colon \mathbf{1} \to \mathcal{C}$ for a given category $\mathcal{C}$ picks out a single object X of $\mathcal{C}$.

- The slice category $(\mathcal{C}/X)$ is the comma category $(\mathrm{Id}_{\mathcal{C}}{\downarrow}\Delta_X)$.
- The coslice category $(X/\mathcal{C})$ is the comma category $(\Delta_X{\downarrow}\mathrm{Id}_{\mathcal{C}})$. $\Diamond$

Objects of a slice category $(\mathcal{C}/X)$ are triples $(A, f, 1)$ where $f\colon A \to X$ and 1 is the single object of $\mathbf{1}$. Similarly, objects of a coslice category $(X/\mathcal{C})$ are triples $(1, f, A)$ with $f\colon X \to A$. Since 1 is the only object of $\mathbf{1}$ we can simply omit it from the triples. Morphisms from (A, f) to (B, g) in the slice category are tuples (h, id_1) with $h\colon A \to B$ a morphism of $\mathcal{C}$. Morphisms from (f, A) to (g, B) in the coslice category are tuples (id_1, h) again with $h\colon A \to B$ of $\mathcal{C}$. As before we can simply omit id_1, and thus morphisms of the slice and coslice categories $(\mathcal{C}/X)$ and $(X/\mathcal{C})$ are just morphisms of $\mathcal{C}$ such that the two equivalent diagrams on the left and respectively the two diagrams on the right commute.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Id}A \arrow[r, "\mathrm{Id}h"] \arrow[d, "f"'] & \mathrm{Id}B \arrow[d, "g"] \\ \Delta_X 1 \arrow[r, "\Delta_X \mathrm{id}_1"'] & \Delta_X 1 \end{tikzcd} \qquad \begin{tikzcd} \Delta_X 1 \arrow[r, "\Delta_X \mathrm{id}_1"] \arrow[d, "f"'] & \Delta_X 1 \arrow[d, "g"] \\ \mathrm{Id}A \arrow[r, "\mathrm{Id}h"'] & \mathrm{Id}B \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "h"'] & & B \end{tikzcd}$$